Math 1433

23 October 2023

What angle $\theta \in [0,180°]$ has $\cos(\theta) = \frac{-1}{2}$?

Answer: 120° or $\frac{2}{3}\pi$

(half) problem session

Topic: equations of lines using vectors

Note: List 2 also has many tasks about planes. We will discuss those next week.

Dot product

The **dot product** of two vectors $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$,

also called the **scalar product** or **inner product**, is written as $\vec{a} \cdot \vec{b}$ (said out loud as “A dot B”). It is a *number* that can be computed as either

- $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

or

- $\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\left(\text{angle between } \vec{a} \text{ and } \vec{b}\right)$.

Last time

Example: Find the angle between $\langle\sqrt{3},1\rangle$ and $\langle 0,7\rangle$.

$\vec{a}$ $\vec{b}$

$|\vec{a}| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$

$|\vec{b}| = \sqrt{0^2 + 7^2} = 7$

So $\vec{a} \cdot \vec{b} = (2)(7)\cos\theta$.

But also $\vec{a} \cdot \vec{b} = (\sqrt{3})(0) + (1)(7) = 7$, so

$(2)(7)\cos\theta = 7 \rightarrow \cos\theta = 1/2 \rightarrow \boxed{\theta = 60^\circ}$

Orthogonal / perpendicular

Two vectors are called **orthogonal** if their dot product is zero.

- For non-zero vectors, this means they are **perpendicular** (or **normal**). Why?

$$\cos(\text{angle}) = 0 \quad \leftrightarrow \quad \text{angle} = 90^\circ$$

- The zero vector is orthogonal to every vector.

Which of $\vec{a} = [2, 1]$ or $\vec{b} = [2, 3]$ is orthogonal to $[2, -4]$?

Give an example of…

- a non-zero vector that is orthogonal to $\vec{w} = [7, 3]$.

  All are scalar multiples of [3, -7].

- a non-zero vector that is orthogonal to $\vec{a} = [3, -1, 2]$.

  There are many totally different answers.

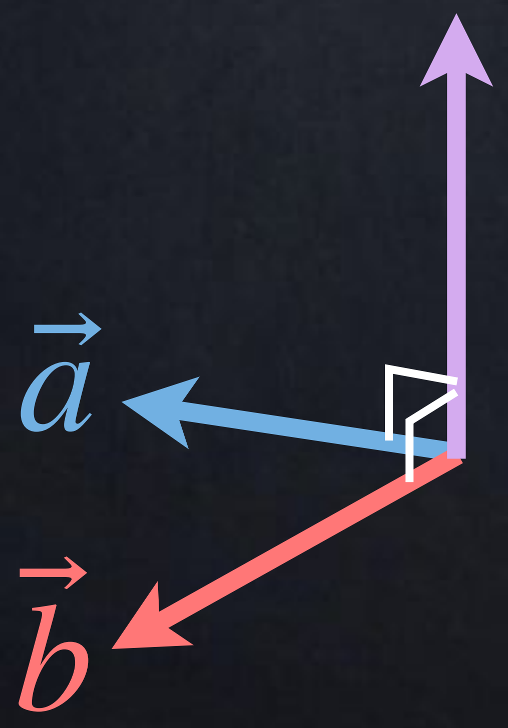
- a non-zero vector that is orthogonal to $\vec{a} = [3, -1, 2]$ and $\vec{b} = [4, 2, 5]$.

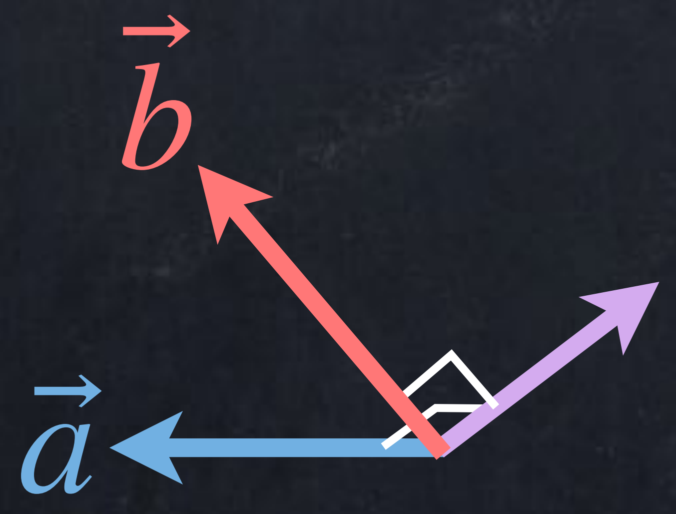
  All are scalar multiples of [9, 7, -10], but it's much harder to see why or to find that vector in the first place.

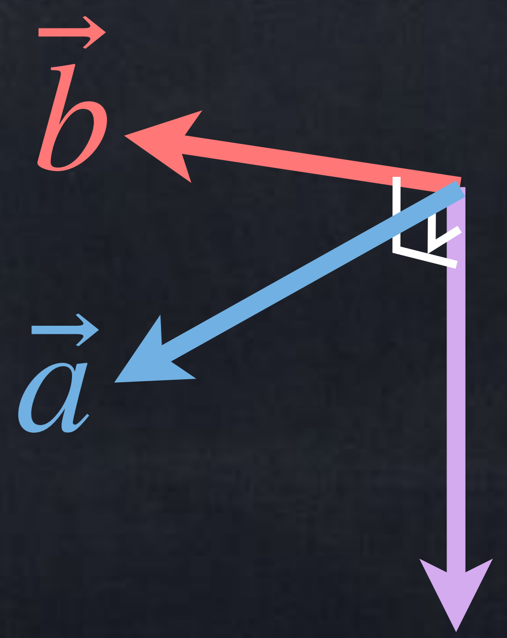
Cross product

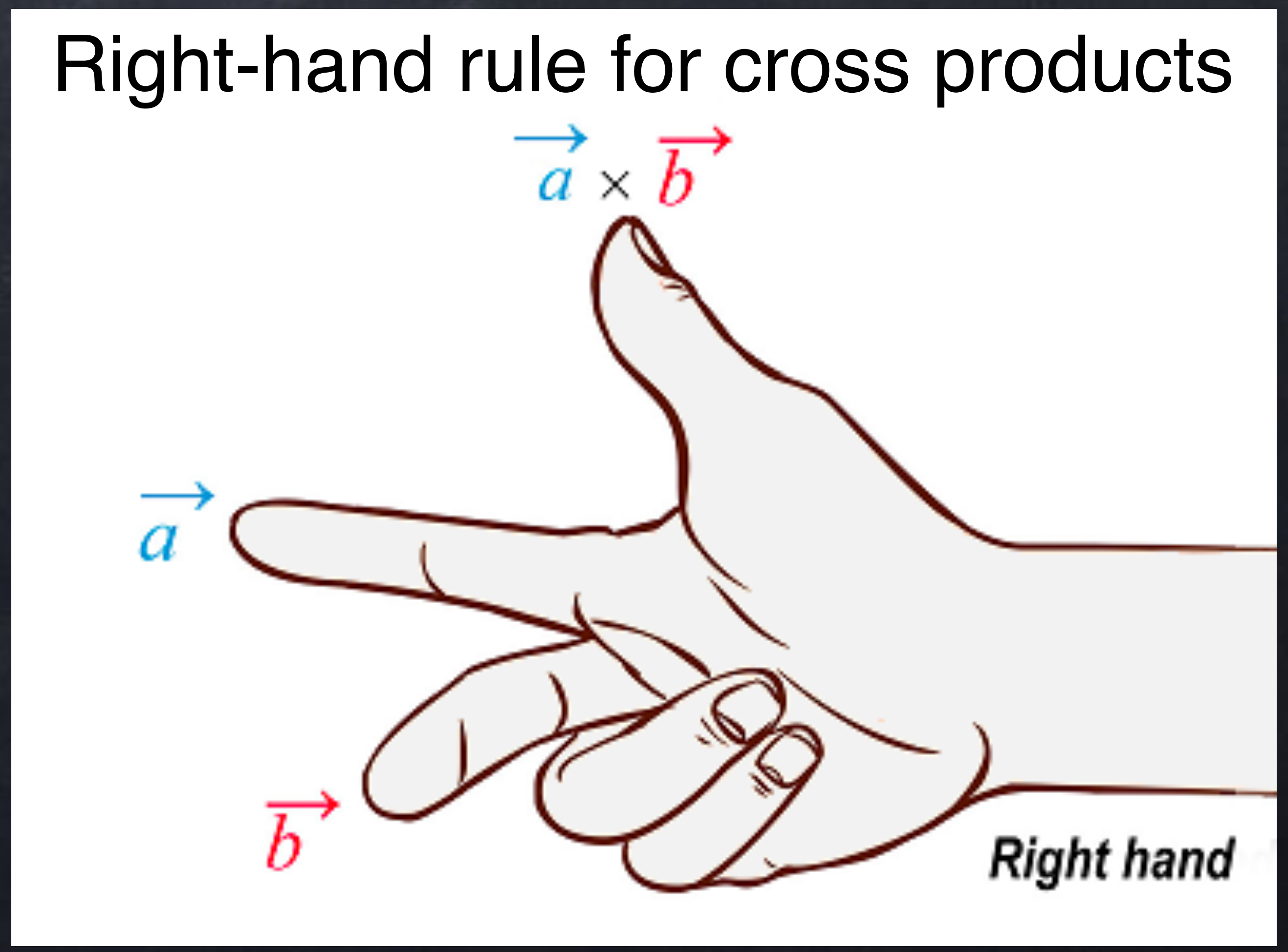
For 3D vectors only, the **cross product of $\vec{a}$ and $\vec{b}$**, written $\vec{a} \times \vec{b}$ and spoken as "A cross B", is the unique vector that is

- perpendicular to both $\vec{a}$ and $\vec{b}$,
- has length $\left|\vec{a}\right| \left|\vec{b}\right| \sin(\theta)$, where θ is the angle between $\vec{a}$ and $\vec{b}$,
- and points in the direction given by the "Right-Hand Rule".









Cross product

For 3D vectors only, the **cross product of $\vec{a}$ and $\vec{b}$**, written $\vec{a} \times \vec{b}$, can be calculated using only

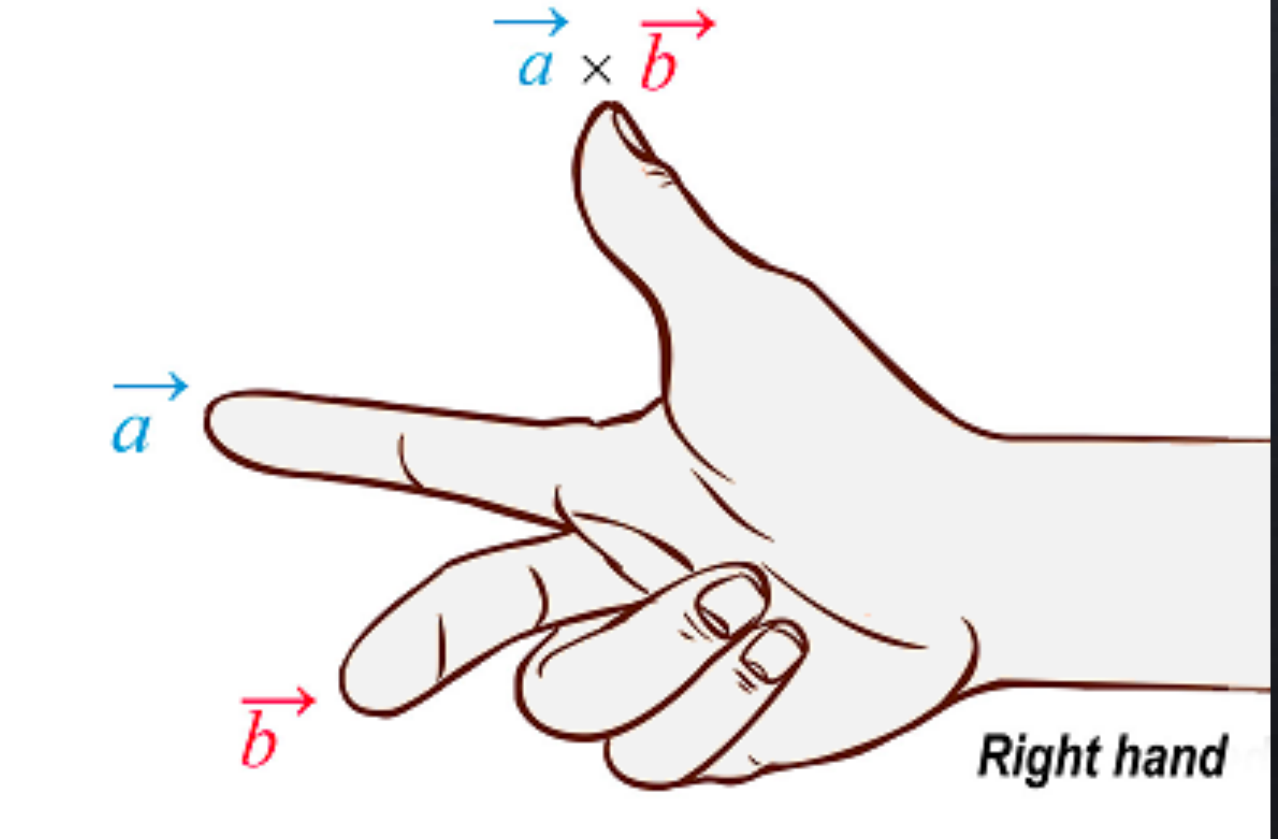
$$\hat{\imath} \times \hat{\jmath} = \hat{k} \qquad \hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \hat{k} \times \hat{\imath} = \hat{\jmath} \qquad \vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$$

and *very* careful algebra.

There is also a direct formula, but it's ugly:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Right-hand rule for cross products



Example: Calculate $[3, -1, 2] \times [4, 2, 5]$.

$$(3\hat{\imath} - \hat{\jmath} + 2\hat{k}) \times (4\hat{\imath} + 2\hat{\jmath} + 5\hat{k})$$

$$= (3\hat{\imath} - \hat{\jmath} + 2\hat{k})\times 4\hat{\imath} + (3\hat{\imath} - \hat{\jmath} + 2\hat{k})\times 2\hat{\jmath} + \cdots$$

$$= 12(\hat{\imath}\times\hat{\imath}) - 4(\hat{\jmath}\times\hat{\imath}) + 8(\hat{k}\times\hat{\imath}) + 6(\hat{\imath}\times\hat{\jmath}) + (-2)(\hat{\jmath}\times\hat{\jmath}) + \cdots\cdots$$

$$= 12(\vec{0}) - 4(-\hat{k}) + 8(\hat{\jmath}) + 6(\hat{k}) + 2(\vec{0}) + \cdots\cdots$$

$$= (4\hat{k} + 8\hat{\jmath}) + (6\hat{k} - 4\hat{\imath}) + (-5\hat{\imath} - 15\hat{\jmath})$$

$$= (-4\hat{\imath} - 5\hat{\imath}) + (8\hat{\jmath} - 15\hat{\jmath}) + (4\hat{k} + 6\hat{k})$$

$$= -9\hat{\imath} - 7\hat{\jmath} + 10\hat{k}$$

Note $\hat{\imath} \times \hat{\jmath} = \hat{k}$

but $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.

The direct formula is faster but requires more memorization.

Obvious (?) formulas:

- $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$
- $\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a-c \\ b-d \end{bmatrix}$
- $s \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} sa \\ sb \end{bmatrix}$

Also $\left|\vec{v}\right| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

We will never use $\begin{bmatrix} ac \\ bd \end{bmatrix}$ at all.

Surprising (?) formulas:

- $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$
- $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$
- $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} bf-ce \\ cd-af \\ ae-bd \end{bmatrix}$

Which of these calculations are possible if $\vec{a} \in \mathbb{R}^2$ and $\vec{b} \in \mathbb{R}^3$?

1. $\vec{a} + \vec{b}$ — No (vectors have different dimensions)
2. $\vec{a} \;+\; \vec{b}$ — Yes (simply adding two numbers)
3. $\vec{a}\vec{b}$ — No
4. $\vec{a} \times \vec{a}$ — No (both vectors must be 3D for ×)
5. $\vec{b} \times \vec{b}$ — Yes (note: it will be [0,0,0])
6. $\vec{a} \;\; \vec{b}$ — Yes (scalar multiplication)
7. $\vec{a} \cdot \vec{a} + \vec{b} \times \vec{b}$ — No

Quiz 2

Date: 30 October

Topics:

- vector subtraction
- magnitude of a vector
- dot product

I will write the formula $\left|\vec{a}\right| \left|\vec{b}\right| \cos\theta$ on the quiz paper for you.

I will *not* give you the formulas $a_1 b_1 + \cdots$ or $\sqrt{a_1^2 + \cdots}$.

In the future, I will give you a formula for $\vec{a} \times \vec{b}$ if you need it.